THE HARDY–LITTLEWOOD–PÓLYA INEQUALITY FOR ANALYTIC FUNCTIONS FROM HARDY–SOBOLEV SPACES

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Abstract. In the paper for a function of complex variable analytic in a strip the extremum of the $L_2(\mathbb{R})$-norm of the k-th derivative is found under the restriction on the $L_2(\mathbb{R})$-norm of the function and the norm of its n-th derivative in the Hardy–Sobolev space metric. The problem which is closely connected with this one of optimal recovery of the k-th derivative of a function from the Hardy–Sobolev class by the trace of this function on the real axis given inaccurately is also studied. An optimal method of recovery is obtained.

1. Statement of the problems

In the book [1] Hardy, Littlewood, and Pólya proved that for all integers $0 < k < r$ the exact inequality

$$\|x^{(k)}(\cdot)\|_{L_2(\mathbb{R})} \le \|x(\cdot)\|_{L_2(\mathbb{R})}^{1-\frac{k}{r}} \|x^{(r)}(\cdot)\|_{L_2(\mathbb{R})}^{\frac{k}{r}} \tag{1}$$

holds for all functions $x(\cdot) \in L_2(\mathbb{R})$ for which the $(r-1)$-st derivative is locally absolute continuous on $\mathbb{R}$ and $x^{(r)}(\cdot) \in L_2(\mathbb{R})$. The inequalities of the form (1) are called *Landau–Kolmogorov type inequalities*: Landau [2] was the first who obtained several exact results in similar inequalities and Kolmogorov [3] in 1939 obtained one of the most remarkable result in this problems (he found the exact constant in the inequality similar to (1) when all the norms are taken in the space $L_\infty(\mathbb{R})$). More detailed information about Landau–Kolmogorov types inequalities for real functions may be found in [4] and [5].

It should be stressed that already in Kolmogorov's paper [3] an interest in inequalities of such type for analytic functions was shown. An analogue of the Kolmogorov inequality for functions analytic in the strip $S_\beta = \{z \in \mathbb{C} : |\operatorname{Im} z| < \beta\}$ was obtained in [6]. In this paper an analogue of the Hardy–Littlewood–Pólya inequality for function analytic in the strip S_β is obtained and a series of optimal recovery problems closely connected with this inequality is also considered. Inequalities for derivatives for functions analytic in the strip S_β are also

The research was carried out with the financial support of the Russian Foundation for Basic Research (grant nos. 05-01-00275 and 05-01-00261) and the Program "Universities of Russia" (grant nos. UR.03.01.130 and UR.04.02.536).

interested since the passage to the limit when $\beta \to 0$ gives exact inequalities for the real case.

Now we proceed to the accurate statement of the problem. The *Hardy space* $\mathcal{H}_2^\beta$ is the set of functions $f(\cdot)$ that are analytic in the strip S_β for which

$$\|f(\cdot)\|_{\mathcal{H}_2^\beta} = \left(\sup_{0\le\eta<\beta} \frac{1}{2}\int_{\mathbb{R}} (|f(t+i\eta)|^2 + |f(t-i\eta)|^2)\,dt\right)^{1/2} < \infty.$$

We denote by $\mathcal{H}_2^{r,\beta}$ (the *Hardy–Sobolev space*) the set of functions analytic in the strip S_β for which $f^{(r)}(\cdot) \in \mathcal{H}_2^\beta$.

By the exact Hardy–Littlewood–Pólya inequality for functions from $\mathcal{H}_2^{r,\beta}$ we mean the problem of finding the value

$$\sup_{\substack{f(\cdot)\in\mathcal{H}_2^{r,\beta}\cap L_2(\mathbb{R})\\ \|f^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta}\le\gamma_1\\ \|f(\cdot)\|_{L_2(\mathbb{R})}\le\gamma_2}} \|f^{(k)}(\cdot)\|_{L_2(\mathbb{R})} \tag{2}$$

for all $\gamma_1, \gamma_2 > 0$. We shall consider a more general problem (about optimal recovery of the k-th derivative of $f(\cdot) \in \mathcal{H}_2^{r,\beta}$ in the $L_2(\mathbb{R})$-metric by inaccurate values of function $f(\cdot)$ on $\mathbb{R}$) solving which we obtain the value (2).

Denote by $H_2^{r,\beta}$ the set of functions $f(\cdot) \in \mathcal{H}_2^{r,\beta}$ for which $\|f^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta} \le 1$. Consider the problem of optimal recovery of the k-th derivative of $f(\cdot) \in H_2^{r,\beta} \cap L_2(\mathbb{R})$ by its trace on $\mathbb{R}$ given with an error in the $L_2(\mathbb{R})$-metric, that is, we assume that instead of the trace of $f(\cdot)$ on $\mathbb{R}$ we know a function $y(\cdot) \in L_2(\mathbb{R})$ such that

$$\|f(\cdot) - y(\cdot)\|_{L_2(\mathbb{R})} \le \delta.$$

The task is to recover $f^{(k)}(\cdot)$ on $\mathbb{R}$ by $y(\cdot)$ in the best way.

Any maps $\varphi\colon L_2(\mathbb{R}) \to L_2(\mathbb{R})$ are admitted as *recovery methods*. For a given recovery method φ the quantity

$$e_k(H_2^{r,\beta}, \delta, \varphi) = \sup_{\substack{f(\cdot)\in H_2^{r,\beta}\cap L_2(\mathbb{R}),\ y(\cdot)\in L_2(\mathbb{R})\\ \|f(\cdot)-y(\cdot)\|_{L_2(\mathbb{R})}\le\delta}} \|f^{(k)}(\cdot) - \varphi(y)(\cdot)\|_{L_2(\mathbb{R})}$$

is called the *error* of the method. The quantity

$$E_k(H_2^{r,\beta}, \delta) = \inf_{\varphi\colon L_2(\mathbb{R})\to L_2(\mathbb{R})} e_k(H_2^{r,\beta}, \delta, \varphi)$$

is called the *error of optimal recovery*, and a method delivering the lower bound is called an *optimal recovery method*.

Consider one more extremal problem: the problem of estimation of $L_2(\mathbb{R})$-norm of $f(\cdot) \in \mathcal{H}_2^\beta$ on the line $\operatorname{Im} z = \rho$, $-\beta < \rho < \beta$, by its $L_2(\mathbb{R})$-norms of boundary values on the lines $\operatorname{Im} z = \pm\beta$ (it is the well-known fact that functions from $\mathcal{H}_2^\beta$ have boundary values almost everywhere on the lines $\operatorname{Im} z = \pm\beta$ which are functions from $L_2(\mathbb{R})$). Thus we mean the extremal problem

$$(3)\quad \|f(\cdot + i\rho)\|_{L_2(\mathbb{R})} \to \max, \quad \|f(\cdot - i\beta)\|_{L_2(\mathbb{R})} \le \delta_1, \quad \|f(\cdot + i\beta)\|_{L_2(\mathbb{R})} \le \delta_2.$$

When norms are taken in the spaces $L_\infty(\mathbb{R})$ and functions are analytic and bounded in a strip the result of such kind is known as Theorem on three lines (see, for example, [7]). The corresponding result for a disk is the Hadamar theorem on three disks [8].

We associate with the problem (3) the problem of optimal recovery of $f(\cdot) \in \mathcal{H}_2^\beta$ on the line $\operatorname{Im} z = \rho$ by its approximate boundary values on the lines $\operatorname{Im} z = \pm\beta$. More precisely, we assume that for every $f(\cdot) \in \mathcal{H}_2^\beta$ we know $y_1(\cdot), y_2(\cdot) \in L_2(\mathbb{R})$ such that

$$\|f(\cdot - i\beta) - y_1(\cdot)\|_{L_2(\mathbb{R})} \le \delta_1, \quad \|f(\cdot + i\beta) - y_2(\cdot)\|_{L_2(\mathbb{R})} \le \delta_2.$$

The task is to recover the function $f(\cdot + i\rho)$ by $y_1(\cdot), y_2(\cdot)$.

Any operators $\varphi\colon L_2(\mathbb{R}) \times L_2(\mathbb{R}) \to L_2(\mathbb{R})$ are admitted as recovery methods. For a given method φ its error is defined by the following equality

$$e_\rho(\mathcal{H}_2^\beta, \delta_1, \delta_2, \varphi) = \sup_{\substack{f(\cdot)\in\mathcal{H}_2^\beta,\ y_1(\cdot),y_2(\cdot)\in L_2(\mathbb{R})\\ \|f(\cdot+i\beta)-y_1(\cdot)\|_{L_2(\mathbb{R})}\le\delta_1\\ \|f(\cdot-i\beta)-y_2(\cdot)\|_{L_2(\mathbb{R})}\le\delta_2}} \|f(\cdot + i\rho) - \varphi(y_1, y_2)(\cdot)\|_{L_2(\mathbb{R})}.$$

The quantity

$$E_\rho(\mathcal{H}_2^\beta, \delta_1, \delta_2) = \inf_{\varphi\colon L_2(\mathbb{R})\times L_2(\mathbb{R})\to L_2(\mathbb{R})} e_\rho(\mathcal{H}_2^\beta, \delta_1, \delta_2, \varphi)$$

is called the error of optimal recovery.

2. Main results

Let $r \in \mathbb{N}$ and β be a positive real number. The function $t^r\sqrt{\cosh 2\beta t}$ is monotonically increase for $t \in \mathbb{R}_+$ from 0 to $+\infty$. Therefore for any $x \in \mathbb{R}_+$ there exists the unique solution of the equation

$$t^r\sqrt{\cosh 2\beta t} = x,$$

which belongs to the interval $[0, +\infty)$. Denote it by $\mu_{r\beta}(x)$.

Theorem 1. *Let $r, k \in \mathbb{N}$ and $k \le r$. For all $\gamma_1, \gamma_2 > 0$ the equality*

$$\sup_{\substack{f(\cdot)\in\mathcal{H}_2^{r,\beta}\cap L_2(\mathbb{R})\\ \|f^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta}\le\gamma_1\\ \|f(\cdot)\|_{L_2(\mathbb{R})}\le\gamma_2}} \|f^{(k)}(\cdot)\|_{L_2(\mathbb{R})} = \gamma_2\mu_{r\beta}^k\left(\frac{\gamma_1}{\gamma_2}\right)$$

holds.

In other words, Theorem 1 states that for all functions from the space $\mathcal{H}_2^{r,\beta}$ which are not equivalent to zero the exact inequality

$$\|f^{(k)}(\cdot)\|_{L_2(\mathbb{R})} \le \|f(\cdot)\|_{L_2(\mathbb{R})}\mu_{r\beta}^k\left(\frac{\|f^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta}}{\|f(\cdot)\|_{L_2(\mathbb{R})}}\right)$$

holds.

Theorem 2. *Let $r, k \in \mathbb{N}$, $k \le r$, and $\delta > 0$. Then for the error of optimal recovery of the k-th derivative the equality*

$$E_k(H_2^{r,\beta}, \delta) = \delta\mu_{r\beta}^k(\delta^{-1})$$

holds, and the method

$$\varphi_0(y)(\cdot) = (\mathcal{K}_{k,\delta}^{r,\beta} * y)(\cdot),$$

where

$$\begin{aligned}(4)\quad &\mathcal{K}_{k,\delta}^{r,\beta}(x)\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}(it)^k\left(1+\frac{k\delta^2t^{2r}\cosh 2\beta t}{r-k+\beta\mu_{r\beta}(\delta^{-1})\tanh(2\beta\mu_{r\beta}(\delta^{-1}))}\right)^{-1}e^{ixt}\,dt,\end{aligned}$$

is an optimal recovery method.

Theorem 3. *For all $-\beta < \rho < \beta$ and $\delta_1, \delta_2 > 0$ the equalities*

$$E_\rho(\mathcal{H}_2^\beta, \delta_1, \delta_2) = \sup_{\substack{f(\cdot)\in\mathcal{H}_2^\beta\\ \|f(\cdot-i\beta)\|_{L_2(\mathbb{R})}\le\delta_1\\ \|f(\cdot+i\beta)\|_{L_2(\mathbb{R})}\le\delta_2}} \|f(\cdot+i\rho)\|_{L_2(\mathbb{R})} = \delta_1^{\frac{\beta-\rho}{2\beta}}\delta_2^{\frac{\beta+\rho}{2\beta}}$$

hold, and the method

$$\begin{aligned}\varphi_0(y_1,y_2)(x) = \delta_2^2(\beta-\rho)&(\mathcal{K}_{\delta_1,\delta_2}^{\rho,\beta} * y_1)(x-i(\beta-\rho))\\ &+\delta_1^2(\beta+\rho)(\mathcal{K}_{\delta_1,\delta_2}^{\rho,\beta} * y_2)(x+i(\beta+\rho)),\end{aligned}$$

where

$$(5)\qquad \mathcal{K}_{\delta_1,\delta_2}^{\rho,\beta}(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{ixt}}{\delta_2^2(\beta-\rho)e^{2\beta t}+\delta_1^2(\beta+\rho)e^{-2\beta t}}\,dt,$$

is optimal.

In particular, it follows by Theorem 3 that for $-\beta < \rho < \beta$ for all $f(\cdot) \in \mathcal{H}_2^\beta$ the exact inequality

$$\|f(\cdot+i\rho)\|_{L_2(\mathbb{R})} \le \|f(\cdot-i\beta)\|_{L_2(\mathbb{R})}^{\frac{\beta-\rho}{2\beta}}\|f(\cdot+i\beta)\|_{L_2(\mathbb{R})}^{\frac{\rho+\beta}{2\beta}}$$

holds.

3. Proofs

To prove Theorems 1 and 2 we need one general result about optimal recovery of linear operators based on the method which was developed in [9], [10]. We proceed to the statement of some general problem of optimal recovery.

Let X be a linear space, $Y_1, \dots, Y_p$ linear spaces with the semi-inner products $(\cdot,\cdot)_{Y_j}$, $j = 1, \dots, p$, and the corresponding semi-norms $\|\cdot\|_{Y_j}$, $j = 1, \dots, p$, $Y_s = L_\infty(\Delta_s)$, $\Delta_s \subseteq \mathbb{R}$, $s = p+1, \dots, m$, $I_j \colon X \to Y_j$, $j = 1, \dots, m$, linear operators, and Z a normed linear space. Consider the problem of optimal recovery of the operator $T \colon X \to Z$ on the set

$$W = \{\, x \in X : \|I_j x\|_{Y_j} \le \delta_j,\ j = 1, \dots, l,\ 0 \le l \le p \,\}$$

by values of operators $I_{l+1}, \dots, I_m$ given with an error (for $l = 0$ we take $W = X$). We assume that for any $x \in W$ we know the vector $y = (y_{l+1}, \dots, y_p, y_{p+1}(\cdot), \dots, y_m(\cdot)) \in Y_{l+1} \times \ldots \times Y_m$ such that $\|I_j x - y_j\|_{Y_j} \le \delta_j$, $j = l+1, \dots, p$, and $|I_s x(t) - y_s(t)| \le \delta_s(t)$ for almost all $t \in \Delta_s$, $s = p+1, \dots, m$ (throughout what follows for functions from $L_\infty(\Delta_s)$ we will not note each time that inequalities hold almost everywhere on Δ_s).

Any operators $\varphi \colon Y_{l+1} \times \ldots \times Y_m \to Z$ are admitted as *recovery methods* of the operator T. For a given method φ the quantity

$$e(T, W, I, \delta, \varphi) = \sup_{\substack{x \in W,\ y \in Y_{l+1} \times \ldots \times Y_m \\ \|I_j x - y_j\|_{Y_j} \le \delta_j,\ j = l+1, \dots, p \\ |I_s x(t) - y_s(t)| \le \delta_s(t),\ s = p+1, \dots, m}} \|Tx - \varphi(y)\|_Z$$

is called the *error of recovery* (here $I = (I_{l+1}, \dots, I_m)$, $\delta = (\delta_{l+1}, \dots, \delta_p, \delta_{p+1}(\cdot), \dots, \delta_m(\cdot))$). The quantity

$$E(T, W, I, \delta) = \inf_{\varphi \colon Y_{l+1} \times \ldots \times Y_m \to Z} e(T, W, I, \delta, \varphi)$$

is called the *error of optimal recovery*, and a method delivering the lower bound is called *optimal*.

The formulated problem is closely connected with the extremal problem

$$(6) \quad \|Tx\|_Z^2 \to \max, \quad \|I_j x\|_{Y_j}^2 \le \delta_j^2,\ j = 1, \dots, p, \quad |I_s x(t)|^2 \le \delta_s^2(t), \quad s = p+1, \dots, m.$$

Denote by $\mathcal{L}(x, \lambda)$ the Lagrange function for this extremal problem

$$\mathcal{L}(x, \lambda) = -\|Tx\|_Z^2 + \sum_{j=1}^{p} \lambda_j \|I_j x\|_{Y_j}^2 + \sum_{s=p+1}^{m} \int_{\Delta_s} \lambda_s(t) |I_s x(t)|^2 \, dt,$$

where $\lambda = (\lambda_1, \dots, \lambda_p, \lambda_{p+1}(\cdot), \dots \lambda_m(\cdot))$, $\lambda_j \ge 0$, $j = 1, \dots, p$, and $\lambda_s(\cdot)$ are measurable nonnegative functions on Δ_s, $s = p+1, \dots, m$.

Theorem 4. *Suppose that there exist measurable nonnegative functions $\widehat{\lambda}_s(\cdot)$, $s=p+1,\dots,m$, on Δ_s and $\widehat{\lambda}_j \ge 0$, $j=1,\dots,p$, such that for $\widehat{\lambda}=(\widehat{\lambda}_1,\dots,\widehat{\lambda}_p,\widehat{\lambda}_{p+1}(\cdot),\dots\widehat{\lambda}_m(\cdot))$*

$$(a)\quad \mathcal{L}(x,\widehat{\lambda}) \ge 0 \quad \forall x \in X.$$

Furthermore, suppose that there exists a sequence $\{x_n\}$ of admissible elements in (6) such that the following conditions hold:

$$(b)\quad \lim_{n\to\infty} \mathcal{L}(x_n,\widehat{\lambda}) = 0,$$

$$(c)\quad \lim_{n\to\infty} \left(\sum_{j=1}^{p} \widehat{\lambda}_j \left(\|I_j x_n\|_{Y_j}^2 - \delta_j^2 \right) + \sum_{s=p+1}^{m} \int_{\Delta_s} \widehat{\lambda}_s(t) \left(|I_s x_n(t)|^2 - \delta_s^2(t) \right) dt \right) = 0.$$

Then the value of the extremal problem (6) is equal to

$$\sum_{j=1}^{p} \widehat{\lambda}_j \delta_j^2 + \sum_{s=p+1}^{m} \int_{\Delta_s} \widehat{\lambda}_s(t) \delta_s^2(t)\, dt.$$

Moreover, if for all $y=(y_{l+1},\dots,y_p,y_{p+1}(\cdot),\dots,y_m(\cdot)) \in Y_{l+1}\times\ldots\times Y_m$ there exists x_y which is a solution of the extremal problem

$$(7)\quad \sum_{j=1}^{l} \widehat{\lambda}_j \|I_j x\|_{Y_j}^2 + \sum_{j=l+1}^{p} \widehat{\lambda}_j \|I_j x - y_j\|_{Y_j}^2 + \sum_{s=p+1}^{m} \int_{\Delta_s} \widehat{\lambda}_s(t) |I_s x(t) - y_s(t)|^2\, dt \to \min, \quad x \in X,$$

then

$$(8)\qquad \varphi_0(y) = T x_y$$

is an optimal method of recovery and

$$(9)\qquad E(T,W,I,\delta) = \sqrt{\sum_{j=1}^{p} \widehat{\lambda}_j \delta_j^2 + \sum_{s=p+1}^{m} \int_{\Delta_s} \widehat{\lambda}_s(t) \delta_s^2(t)\, dt}.$$

Proof. Let us show that the values of the problems (6) and

$$(10)\quad \|Tx\|_Z^2 \to \max, \quad \sum_{j=1}^{p} \widehat{\lambda}_j \|I_j x\|_{Y_j}^2 + \sum_{s=p+1}^{m} \int_{\Delta_s} \widehat{\lambda}_s(t) |I_s x(t)|^2\, dt \le S,$$

where

$$S = \sum_{j=1}^{p} \widehat{\lambda}_j \delta_j^2 + \sum_{s=p+1}^{m} \int_{\Delta_s} \widehat{\lambda}_s(t) \delta_s^2(t)\, dt,$$

coincide and are equal to S. Indeed, for any admissible element $x \in X$ in (6) or in (10) with regard to (a) we have

$$-\|Tx\|_Z^2 \geq -\|Tx\|_Z^2 + \sum_{j=1}^{p} \widehat{\lambda}_j \left(\|I_j x\|_{Y_j}^2 - \delta_j^2\right) + \sum_{s=p+1}^{m} \int_{\Delta_s} \widehat{\lambda}_s(t) \left(|I_s x(t)|^2 - \delta_s^2(t)\right) dt \geq -S.$$

On the other hand, using (c) and (b), we obtain

$$-\lim_{n\to\infty} \|Tx_n\|^2 = \lim_{n\to\infty} \Bigg(-\|Tx_n\|_Z^2 + \sum_{j=1}^{p} \widehat{\lambda}_j \left(\|I_j x_n\|_{Y_j}^2 - \delta_j^2\right) + \sum_{s=p+1}^{m} \int_{\Delta_s} \widehat{\lambda}_s(t) \left(|I_s x_n(t)|^2 - \delta_s^2(t)\right) dt \Bigg) = -S,$$

that is, S is the value of problems (6) and (10).

The lower bound. For any method φ for all $x \in W$ such that $\|I_j x\|_{Y_j} \leq \delta_j$, $j = l+1, \ldots, p$, and $|I_s x(t)| \leq \delta_s(t)$, $s = p+1, \ldots, m$, we have

$$2\|Tx\|_Z \leq \|Tx - \varphi(0)\|_Z + \|T(-x) - \varphi(0)\|_Z \leq 2e(T, W, I, \delta, \varphi).$$

Consequently, for any method φ

$$e(T, W, I, \delta, \varphi) \geq \sup_{\substack{x\in W \\ \|I_j x\|_{Y_j} \leq \delta_j,\ j=l+1,\ldots,p \\ |I_s x(t)| \leq \delta_s(t),\ s=p+1,\ldots,m}} \|Tx\|_Z = \sqrt{S}.$$

Thus,

$$E(T, W, I, \delta) \geq \sqrt{S}. \tag{11}$$

The upper bound. Consider the linear space $E = Y_1 \times \ldots \times Y_m$ with the semi-inner product

$$(y^1, y^2)_E = \sum_{j=1}^{p} \widehat{\lambda}_j (y_j^1, y_j^2)_{Y_j} + \sum_{s=p+1}^{m} \int_{\Delta_j} \widehat{\lambda}_s(t) y_j^1(t) \overline{y_j^2(t)}\, dt.$$

Then the extremal problem (7) can be rewritten in the form

$$\|\widetilde{I}x - \widehat{y}\|_E^2 \to \min, \quad x \in X,$$

where $\widetilde{I} = (I_1, \ldots, I_m)$ and $\widehat{y} = (0, \ldots, 0, y_{l+1}, \ldots, y_p, y_{p+1}(\cdot), \ldots, y_m(\cdot))$. If x_y is a solution of this problem, then it is easily seen that for all $x \in X$ the equality

$$(\widetilde{I}x_y - \widehat{y}, \widetilde{I}x)_E = 0$$

holds. It follows that

$$\|\widetilde{I}x - \widehat{y}\|_E^2 = \|\widetilde{I}x - \widetilde{I}x_y\|_E^2 + \|\widetilde{I}x_y - \widehat{y}\|_E^2. \tag{12}$$

If $x \in W$ and $\widehat{y}$ such that $\|I_jx - y_j\|_{Y_j} \le \delta_j$, $j = l+1, \dots, p$, $|I_sx(t) - y_s(t)| \le \delta_s(t)$ almost everywhere on Δ_s, $s = p+1, \dots, m$, then it follows from (12) that

$$\|\widetilde{I}x - \widetilde{I}x_y\|_E^2 \le \|\widetilde{I}x - \widehat{y}\|_E^2 = \sum_{j=1}^{l} \widehat{\lambda}_j \|I_jx\|_{Y_j}^2 + \sum_{j=l+1}^{p} \widehat{\lambda}_j \|I_jx - y_j\|_{Y_j}^2 + \sum_{s=p+1}^{m} \int_{\Delta_s} \widehat{\lambda}_s(t) |I_sx(t) - y_s(t)|^2 \, dt \le S.$$

By setting $z = x - x_y$, we arrive at the inequality

$$\sum_{j=1}^{p} \widehat{\lambda}_j \|I_jz\|_{Y_j}^2 + \sum_{s=p+1}^{m} \int_{\Delta_s} \widehat{\lambda}_s(t) |I_sz(t)|^2 \, dt \le S.$$

Thus for the method (8) we have

$$\begin{aligned} \|Tx - \varphi_0(y)\|_Z &= \|Tz\|_Z \\ &\le \sup\left\{ \|Tx\|_Z : \sum_{j=1}^{p} \widehat{\lambda}_j \|I_jx\|_{Y_j}^2 + \sum_{s=p+1}^{m} \int_{\Delta_s} \widehat{\lambda}_s(t) |I_sx(t)|^2 \, dt \le S \right\} \\ &= \sqrt{S}. \end{aligned}$$

Taking account of (11), we obtain the equality (9) and prove the optimality of the method (8). □

Proof of Theorem 1. Consider the extremal problem

(13) $$\|f^{(k)}(\cdot)\|_{L_2(\mathbb{R})}^2 \to \max, \quad \|f^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta}^2 \le \gamma_1^2, \quad \|f(\cdot)\|_{L_2(\mathbb{R})}^2 \le \gamma_2^2.$$

For this problem the Lagrange function has the form

$$\mathcal{L}(f(\cdot), \lambda_1, \lambda_2) = -\|f^{(k)}(\cdot)\|_{L_2(\mathbb{R})}^2 + \lambda_1 \|f^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta}^2 + \lambda_2 \|f(\cdot)\|_{L_2(\mathbb{R})}^2.$$

It follows by the basic theorem about the representation of functions from spaces $\mathcal{H}_2$ in tubular domains (see [7]) that $f(\cdot) \in \mathcal{H}_2^\beta$ iff it has the form

(14) $$f(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(t) e^{izt} \, dt,$$

where $\widehat{f}(\cdot)$ is a function satisfying the condition

$$\sup_{|y|<\beta} \int_{\mathbb{R}} |\widehat{f}(t)|^2 e^{-2yt} \, dt < \infty$$

($\widehat{f}(\cdot)$ is the Fourier transform of $f(x)$, $x \in \mathbb{R}$). Then it follows by the Plancherel theorem that

$$\|f(\cdot)\|_{\mathcal{H}_2^\beta}^2 = \frac{1}{2\pi} \sup_{0 \le y < \beta} \int_{\mathbb{R}} |\widehat{f}(t)|^2 \cosh 2yt \, dt = \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{f}(t)|^2 \cosh 2\beta t \, dt.$$

Passing to Fourier transforms and writing $(2\pi)^{-1}|\widehat{f}(\cdot)|^2 = u(\cdot)$, we have

$$\mathcal{L}(f(\cdot), \lambda_1, \lambda_2) = \int_{\mathbb{R}} (-t^{2k} + \lambda_1 t^{2r} \cosh 2\beta t + \lambda_2) u(t)\, dt.$$

Set

$$\alpha(t) = -1 + \lambda_1 t^{2(r-k)} \cosh 2\beta t + \lambda_2 t^{-2k}.$$

It is easy to verify that $\alpha(t)$ is a convex function for $t > 0$. Therefore, if at some point $t_0 > 0$ the equalities

$$\alpha(t_0) = \alpha'(t_0) = 0 \tag{15}$$

hold, then $\alpha(t) \ge 0$ for all $t \ge 0$. Let t_0 be a positive solution of the equation

$$\gamma_2^2 t^{2r} \cosh 2\beta t = \gamma_1^2, \tag{16}$$

that is, $t_0 = \mu_{r\beta}(\gamma_1/\gamma_2)$. We take $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ such that equalities (15) are fulfilled. We have

$$\widehat{\lambda}_1 = \frac{k t_0^{2(k-r)}}{r \cosh 2\beta t_0 + t_0 \beta \sinh 2\beta t_0},$$

$$\widehat{\lambda}_2 = \frac{(r-k) t_0^{2k} \cosh 2\beta t_0 + t_0^{2k+1} \beta \sinh 2\beta t_0}{r \cosh 2\beta t_0 + t_0 \beta \sinh 2\beta t_0}.$$

Thus, for the chosen $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ for all $t \in \mathbb{R}$

$$-t^{2k} + \widehat{\lambda}_1 t^{2r} \cosh 2\beta t + \widehat{\lambda}_2 \ge 0.$$

Consequently, for all $f(\cdot) \in \mathcal{H}_2^{r,\beta} \cap L_2(\mathbb{R})$

$$\mathcal{L}(f(\cdot), \widehat{\lambda}_1, \widehat{\lambda}_2) \ge 0.$$

For sufficiently big n (such that $t_0 - 1/n > 0$) we set

$$u_n(t) = \begin{cases} \gamma_2^2 n, & t \in (t_0 - 1/n, t_0), \\ 0, & t \notin (t_0 - 1/n, t_0). \end{cases}$$

Denote by $f_n(\cdot)$ the sequence of functions obtained by (14) for $\widehat{f}(\cdot) = \sqrt{2\pi u_n(\cdot)}$. We have

$$\|f_n(\cdot)\|^2_{L_2(\mathbb{R})} = \|u_n(\cdot)\|^2_{L_2(\mathbb{R})} = \gamma_2^2$$

and

$$\|f_n^{(r)}(\cdot)\|^2_{L_2(\mathbb{R})} = \gamma_2^2 n \int_{t_0 - 1/n}^{t_0} t^{2r} \cosh 2\beta t\, dt \le \gamma_2^2 t_0^{2r} \cosh 2\beta t_0 = \gamma_1^2,$$

that is, the functions $f_n(\cdot)$ are admissible in the problem (13). It is easy to verify that

$$\lim_{n\to\infty} \mathcal{L}(f_n(\cdot), \widehat{\lambda}_1, \widehat{\lambda}_2) = 0, \tag{17}$$

$$\lim_{n\to\infty} \|f_n^{(r)}(\cdot)\|^2_{L_2(\mathbb{R})} = \gamma_1^2. \tag{18}$$

It follows by Theorem 4 that the value of the extremal problem (13) is equal to

$$\widehat{\lambda}_1\gamma_1^2+\widehat{\lambda}_2\gamma_2^2=\gamma_2^2\mu_{r\beta}^{2k}\left(\frac{\gamma_1}{\gamma_2}\right).$$

□

Proof of Theorem 2. Set $\gamma_1=1$, $\gamma_2=\delta$ and for the corresponding $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ for some $y(\cdot)\in L_2(\mathbb{R})$ consider the extremal problem

$$\widehat{\lambda}_1\|f^{(r)}(\cdot)\|^2_{\mathcal{H}_2^\beta}+\widehat{\lambda}_2\|f(\cdot)-y(\cdot)\|^2_{L_2(\mathbb{R})}\to\min,\quad f(\cdot)\in\mathcal{H}_2^{r,\beta}\cap L_2(\mathbb{R}).$$

Passing to Fourier transforms this problem can be rewritten in the form

$$\frac{1}{2\pi}\int_{\mathbb{R}}(\widehat{\lambda}_1t^{2r}|\widehat{f}(t)|^2\cosh 2\beta t+\widehat{\lambda}_2|\widehat{f}(\cdot)-\widehat{y}(\cdot)|^2)\,dt\to\min,$$
$$f(\cdot)\in\mathcal{H}_2^{r,\beta}\cap L_2(\mathbb{R}).$$

It is easy to show that

$$\widehat{f}_y(t)=\frac{\widehat{\lambda}_2}{\widehat{\lambda}_2+\widehat{\lambda}_1t^{2r}\cosh 2\beta t}\widehat{y}(t)$$

is the solution of this problem. I follows by Theorem 4 that the method

$$\varphi_0(y)(\cdot)=f_y^{(k)}(\cdot)$$

is optimal. Since the Fourier transform of $f_y^{(k)}(\cdot)$ equals

$$(it)^k\frac{\widehat{\lambda}_2}{\widehat{\lambda}_2+\widehat{\lambda}_1t^{2r}\cosh 2\beta t}\widehat{y}(t),$$

it can be written in the form of convolution of $y(\cdot)$ and $\mathcal{K}_{k,\delta}^{r,\beta}(\cdot)$ which is defined by (4). □

Proof of Theorem 3. For convenience we pass to squares in the extremal problem (3)

$$\|f(\cdot+i\rho)\|^2_{L_2(\mathbb{R})}\to\max,\quad \|f(\cdot-i\beta)\|^2_{L_2(\mathbb{R})}\le\delta_1^2,$$
$$\|f(\cdot+i\beta)\|^2_{L_2(\mathbb{R})}\le\delta_2^2. \tag{19}$$

and write the corresponding Lagrange function

$$\mathcal{L}(f(\cdot),\lambda_1,\lambda_2)=-\|f(\cdot+i\rho)\|^2_{L_2(\mathbb{R})}+\lambda_1\|f(\cdot-i\beta)\|^2_{L_2(\mathbb{R})}$$
$$+\lambda_2\|f(\cdot+i\beta)\|^2_{L_2(\mathbb{R})}.$$

In view of the representation (14) passing to Fourier transforms by the Plancherel theorem we have

$$\mathcal{L}(f(\cdot),\lambda_1,\lambda_2)=\frac{1}{2\pi}\int_{\mathbb{R}}(-e^{-2\rho t}+\lambda_1e^{2\beta t}+\lambda_2e^{-2\beta t})|\widehat{f}(t)|^2\,dt.$$

It is easy to verify that the function

$$\gamma(t)=-1+\lambda_1e^{2(\beta+\rho)t}+\lambda_2e^{-2(\beta-\rho)t}$$

is convex on $\mathbb{R}$. Therefore if at some point $t_0 \in \mathbb{R}$

$$\gamma(t_0) = \gamma'(t_0) = 0, \tag{20}$$

then $\gamma(t) \geq 0$ for all $t \in \mathbb{R}$. We set

$$t_0 = \frac{\ln \delta_1 - \ln \delta_2}{2\beta}$$

and take $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ from the condition (20). We have

$$\widehat{\lambda}_1 = \frac{\beta - \rho}{2\beta} e^{-2(\beta+\rho)t_0}, \quad \widehat{\lambda}_2 = \frac{\beta + \rho}{2\beta} e^{2(\beta-\rho)t_0}.$$

Thus for all $f(\cdot) \in \mathcal{H}_2^\beta$

$$\mathcal{L}(f(\cdot), \widehat{\lambda}_1, \widehat{\lambda}_2) \geq 0.$$

For sufficiently big n denote by $f_n(\cdot)$ the sequence of functions obtained by (14) for

$$\widehat{f}_n(t) = \begin{cases} A_n, & t \in (t_0 - 1/n, t_0 + 1/n), \\ 0, & t \notin (t_0 - 1/n, t_0 + 1/n), \end{cases}$$

where

$$A_n = n\sqrt{\frac{\pi\delta_1\delta_2}{n+1}}.$$

Direct calculations show that

$$\begin{aligned} \|f_n(\cdot - i\beta)\|^2_{L_2(\mathbb{R})} &= \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f}_n(t)|^2 e^{2\beta t}\,dt = \delta_1^2 \frac{n}{n+1}\frac{\sinh 2\beta/n}{2\beta/n}, \\ \|f_n(\cdot + i\beta)\|^2_{L_2(\mathbb{R})} &= \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f}_n(t)|^2 e^{-2\beta t}\,dt = \delta_2^2 \frac{n}{n+1}\frac{\sinh 2\beta/n}{2\beta/n}. \end{aligned}$$

In view of the fact that

$$\frac{\sinh 2\beta/n}{2\beta/n} < 1 + 1/n,$$

the sequence $\widehat{f}_n(\cdot)$ is admissible in the problem (19) for sufficiently large n and moreover

$$\begin{aligned} \lim_{n\to\infty} \|f_n(\cdot - i\beta)\|^2_{L_2(\mathbb{R})} &= \delta_1^2, \\ \lim_{n\to\infty} \|f_n(\cdot + i\beta)\|^2_{L_2(\mathbb{R})} &= \delta_2^2. \end{aligned}$$

It is easy to verify that

$$\lim_{n\to\infty} \mathcal{L}(f_n(\cdot), \widehat{\lambda}_1, \widehat{\lambda}_2) = 0.$$

Thus it follows by Theorem 4 that the value of the extremal problem (19) is equal to

$$\widehat{\lambda}_1\delta_1^2 + \widehat{\lambda}_2\delta_2^2 = \delta_1^{\frac{\beta-\rho}{\beta}}\delta_2^{\frac{\beta+\rho}{\beta}}.$$

Consider now the extremal problem

$$\widehat{\lambda}_1\|f(\cdot - i\beta) - y_1(\cdot)\|^2_{L_2(\mathbb{R})} + \widehat{\lambda}_2\|f(\cdot + i\beta) - y_2(\cdot)\|^2_{L_2(\mathbb{R})} \to \min, \quad f(\cdot) \in \mathcal{H}_2^\beta.$$

Passing to Fourier transforms and using (14) this problem is rewritten in the form

$$\frac{1}{2\pi}\int_{\mathbb{R}}\left(\widehat{\lambda}_1|\widehat{f}(t)e^{\beta t}-\widehat{y}_1(t)|^2+\widehat{\lambda}_2|\widehat{f}(t)e^{-\beta t}-\widehat{y}_2(t)|^2\right)\,dt\to\min,\qquad f(\cdot)\in\mathcal{H}_2^{\beta}.$$

It is easy to verify that the function $f_{y_1,y_2}(\cdot)$ such that

$$\widehat{f}_{y_1,y_2}(t)=\frac{\widehat{\lambda}_1e^{\beta t}\widehat{y}_1(t)+\widehat{\lambda}_2e^{-\beta t}\widehat{y}_2(t)}{\widehat{\lambda}_1e^{2\beta t}+\widehat{\lambda}_2e^{-2\beta t}}$$

is the solution of this problem.

It follows by Theorem 4 that the method

$$\varphi_0(y_1,y_2)(\cdot)=f_{y_1,y_2}(\cdot+i\rho)$$

is optimal. We have

$$\begin{aligned}\varphi_0(y_1,y_2)(x)&\\ &=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{\delta_2^2(\beta-\rho)e^{\beta t}y_1(t)+\delta_1^2(\beta+\rho)e^{-\beta t}y_2(t)}{\delta_2^2(\beta-\rho)e^{2\beta t}+\delta_1^2(\beta+\rho)e^{-2\beta t}}e^{i(x+i\rho)t}\,dt\\ &=\delta_2^2(\beta-\rho)(\mathcal{K}_{\delta_1,\delta_2}^{\rho,\beta}*y_1)(x-i(\beta-\rho))+\delta_1^2(\beta+\rho)(\mathcal{K}_{\delta_1,\delta_2}^{\rho,\beta}*y_2)(x+i(\beta+\rho)),\end{aligned}$$

where the kernel $\mathcal{K}_{\delta_1,\delta_2}^{\rho,\beta}(\cdot)$ is defined by (5). $\square$

4. Further results

4.1. **Recovery of the k-th derivative on the whole strip.** Inequalities for derivatives of analytic functions and recovery problems connected with them are in certain sense an intermediate case between one-dimensional and many-dimensional problems. Nevertheless already here there is some variety connected, in particular, with the fact that the optimal recovery can be considered on various domains (however just as the assignment of input information about a function). As an example consider the optimal recovery problem of the k-th derivative of $f(\cdot)\in H_2^{r,\beta}\cap L_2(\mathbb{R})$ on the whole strip S_β by its trace on $\mathbb{R}$ given with some error in the metric of $L_2(\mathbb{R})$.

Any operators $\varphi\colon L_2(\mathbb{R})\to\mathcal{H}_2^{\beta}$ are admitted as recovery methods. For a given recovery method φ the quantity

$$e_k(H_2^{r,\beta},\delta,S_\beta,\varphi)=\sup_{\substack{f(\cdot)\in H_2^{r,\beta}\cap L_2(\mathbb{R}),\ y(\cdot)\in L_2(\mathbb{R})\\ \|f(\cdot)-y(\cdot)\|_{L_2(\mathbb{R})}\le\delta}}\|f^{(k)}(\cdot)-\varphi(y)(\cdot)\|_{\mathcal{H}_2^{\beta}}$$

is called the error of the method. The quantity

$$E_k(H_2^{r,\beta},\delta,S_\beta)=\inf_{\varphi\colon L_2(\mathbb{R})\to\mathcal{H}_2^{\beta}}e_k(H_2^{r,\beta},\delta,S_\beta,\varphi)$$

is called the error of optimal recovery, and a method delivering the lower bound is called the optimal recovery method.

The corresponding problem about the exact inequality has the form

$$\sup_{\substack{f(\cdot)\in\mathcal{H}_2^{r,\beta}\cap L_2(\mathbb{R})\\ \|f^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta}\le\gamma_1\\ \|f(\cdot)\|_{L_2(\mathbb{R})}\le\gamma_2}} \|f^{(k)}(\cdot)\|_{\mathcal{H}_2^\beta} \tag{21}$$

where $\gamma_1, \gamma_2 > 0$.

Theorem 5. *Let $r, k \in \mathbb{N}$, $k < r$, and $\gamma_1, \gamma_2, \delta > 0$. Then*

$$\sup_{\substack{f(\cdot)\in\mathcal{H}_2^{r,\beta}\cap L_2(\mathbb{R})\\ \|f^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta}\le\gamma_1\\ \|f(\cdot)\|_{L_2(\mathbb{R})}\le\gamma_2}} \|f^{(k)}(\cdot)\|_{\mathcal{H}_2^\beta} = \gamma_1\mu_{r\beta}^{k-r}\left(\frac{\gamma_1}{\gamma_2}\right),$$

$$E_k(H_2^{r,\beta}, \delta, S_\beta) = \mu_{r\beta}^{k-r}(\delta^{-1}),$$

and the method

$$\varphi_0(y)(\cdot) = (\mathcal{M}_{k,\delta}^{r,\beta} * y)(\cdot),$$

where

$$\mathcal{M}_{k,\delta}^{r,\beta}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} (it)^k \left(1 + \frac{\delta^2 t_0^{2(r-k)}}{r-k}(k + \beta t_0\tanh(2\beta t_0))t^{2r}\cosh 2\beta t\right)^{-1} e^{ixt}\,dt, \tag{22}$$

$$t_0 = \mu_{r\beta}(\delta^{-1}),$$

is an optimal method of recovery.

Proof. Consider the extremal problem

$$\|f^{(k)}(\cdot)\|^2_{\mathcal{H}_2^\beta} \to \max, \quad \|f^{(r)}(\cdot)\|^2_{\mathcal{H}_2^\beta} \le \gamma_1^2, \quad \|f(\cdot)\|^2_{L_2(\mathbb{R})} \le \gamma_2^2. \tag{23}$$

For this problem the Lagrange function has the form

$$\mathcal{L}(f(\cdot), \lambda_1, \lambda_2) = -\|f^{(k)}(\cdot)\|^2_{\mathcal{H}_2^\beta} + \lambda_1\|f^{(r)}(\cdot)\|^2_{\mathcal{H}_2^\beta} + \lambda_2\|f(\cdot)\|^2_{L_2(\mathbb{R})}. \tag{24}$$

Passing to Fourier transforms (using the Plancherel theorem and representation (14)) we have

$$\mathcal{L}(f(\cdot), \lambda_1, \lambda_2) = \int_{\mathbb{R}} (-t^{2k}\cosh 2\beta t + \lambda_1 t^{2r}\cosh 2\beta t + \lambda_2)u(t)\,dt,$$

where $u(\cdot) = (2\pi)^{-1}|\widehat{f}(\cdot)|^2$.

Set

$$\omega(t) = -1 + \lambda_1 t^{2(r-k)} + \lambda_2 t^{-2k}\cosh^{-1} 2\beta t.$$

It is easy to show that $\omega(t)$ is a convex function for $t > 0$. Therefore if at some point $t_0 > 0$ the equalities

$$\omega(t_0) = \omega'(t_0) = 0 \tag{25}$$

hold, then $\omega(t)\geq 0$ for all $t\geq 0$. Let t_0 be a positive solution of (16), that is, $t_0=\mu_{r\beta}(\gamma_1/\gamma_2)$. We take $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ such that equalities (25) are fulfilled. We have

$$\widehat{\lambda}_1=\frac{k\cosh 2\beta t_0+t_0\beta\sinh 2\beta t_0}{t_0^{2(r-k)}(k\cosh 2\beta t_0+t_0\beta\sinh 2\beta t_0)+(r-k)\cosh 2\beta t_0},$$
$$\widehat{\lambda}_2=\frac{(r-k)t_0^{2k}\cosh^2 2\beta t_0}{t_0^{2(r-k)}(k\cosh 2\beta t_0+t_0\beta\sinh 2\beta t_0)+(r-k)\cosh 2\beta t_0}.$$

Thus, for the chosen $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ for all $t\in\mathbb{R}$

$$-t^{2k}\cosh 2\beta t+\widehat{\lambda}_1t^{2r}\cosh 2\beta t+\widehat{\lambda}_2\geq 0.$$

Consequently, for all $f(\cdot)\in\mathcal{H}_2^{r,\beta}\cap L_2(\mathbb{R})$

$$\mathcal{L}(f(\cdot),\widehat{\lambda}_1,\widehat{\lambda}_2)\geq 0.$$

The functions $f_n(\cdot)$ defined in the proof of Theorem 1 are admissible in the problem (23), too. It follows from (18) and (17) which is fulfilled for the Lagrange function (24) that the value of the extremal problem (23) is equal to

$$\widehat{\lambda}_1\gamma_1^2+\widehat{\lambda}_2\gamma_2^2=\gamma_1\mu_{r\beta}^{2(k-r)}\left(\frac{\gamma_1}{\gamma_2}\right).$$

The error of optimal recovery and optimal method are obtained by the same scheme which was used in the proof of Theorem 1. □

In particular, it follows by Theorem 5 that for all functions from the space $\mathcal{H}_2^{r,\beta}$ that are not equivalent to zero the exact inequality

$$\|f^{(k)}(\cdot)\|_{\mathcal{H}_2^\beta}\leq\|f^{(r)}(\cdot)\|_{L_2(\mathbb{R})}\mu_{r\beta}^{k-r}\left(\frac{\|f^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta}}{\|f(\cdot)\|_{L_2(\mathbb{R})}}\right)$$

holds

4.2. **Recovery of the k-th derivative by the inaccurate Fourier transform.** Denote by $\mathcal{H}_{2,\infty}^{r,\beta}$ the set of functions $f(\cdot)\in\mathcal{H}_2^{r,\beta}\cap L_2(\mathbb{R})$ for which $\widehat{f}(\cdot)\in L_\infty(\mathbb{R})$ (we denote as before by $\widehat{f}(\cdot)$ the Fourier transform of $f(x)$, $x\in\mathbb{R}$). Denote by $H_{2,\infty}^{r,\beta}$ the set of functions $\mathcal{H}_{2,\infty}^{r,\beta}\cap H_2^{r,\beta}$. Consider the problem of optimal recovery of the k-th derivative of $f(\cdot)\in H_{2,\infty}^{r,\beta}$ by its Fourier transform $\widehat{f}(\cdot)$ given with an error in the metric of $L_\infty(\Delta_\sigma)$ where $\Delta_\sigma=(-\sigma,\sigma)$, $0<\sigma\leq\infty$, that is, we assume that instead of function $\widehat{f}(\cdot)$ we know $y(\cdot)\in L_\infty(\Delta_\sigma)$ such that

$$\|\widehat{f}(\cdot)-y(\cdot)\|_{L_\infty(\Delta_\sigma)}\leq\delta.$$

The task is to recover $f^{(k)}(\cdot)$ on $\mathbb{R}$ in the best way knowing the function $y(\cdot)$.

According to the general statement of the recovery problem of operators the quantity

$$e_k(H^{r,\beta}_{2,\infty},\delta,\sigma,\varphi)=\sup_{\substack{f(\cdot)\in H^{r,\beta}_{2,\infty},\ y(\cdot)\in L_\infty(\Delta_\sigma)\\ \|f(\cdot)-y(\cdot)\|_{L_\infty(\Delta_\sigma)}\le\delta}}\|f^{(k)}(\cdot)-\varphi(y)(\cdot)\|_{L_2(\mathbb{R})}$$

is the error of the recovery method $\varphi\colon L_\infty(\Delta_\sigma)\to L_2(\mathbb{R})$, the quantity

$$E_k(H^{r,\beta}_{2,\infty},\delta,\sigma)=\inf_{\varphi\colon L_\infty(\Delta_\sigma)\to L_2(\mathbb{R})}e_k(H^{r,\beta}_{2,\infty},\delta,\sigma,\varphi)$$

is the error of optimal recovery, and method delivering the lower bound is optimal method of recovery.

Set

$$I_{r,\beta}(\sigma)=\frac{1}{2\pi}\int_{-\sigma}^{\sigma}t^{2r}\cosh 2\beta t\,dt.$$

It follows from the monotonically increase of the function $I_{r,\beta}(\sigma)$, $\sigma\in(0,+\infty)$, and the fact that $I_{r,\beta}(0)=0$ and $I_{r,\beta}(\sigma)\to+\infty$ as $\sigma\to+\infty$ that for all $\delta>0$ there exists the unique $\widehat{\sigma}\in(0,+\infty)$ for which

$$I_{r,\beta}(\widehat{\sigma})=\delta^{-2}. \tag{26}$$

Theorem 6. *Let $r,k\in\mathbb{N}$, $k\le r$, $\delta>0$, and $\widehat{\sigma}$ be defined by (26). Then*

$$E_k(H^{r,\beta}_{2,\infty},\delta,\sigma)=\begin{cases}\sqrt{\dfrac{\sigma^{-2(r-k)}}{\cosh 2\beta\sigma}\Big(1-\delta^2 I_{r,\beta}(\sigma)\Big)+\dfrac{\delta^2\sigma^{2k+1}}{\pi(2k+1)}}, & \sigma<\widehat{\sigma},\\[2ex] \dfrac{\delta\widehat{\sigma}^{k+1/2}}{\sqrt{\pi(2k+1)}}, & \sigma\ge\widehat{\sigma},\end{cases},$$

and the method

$$\varphi_0(y)(\cdot)=\frac{1}{2\pi}\int_{-\sigma_0}^{\sigma_0}(i\tau)^k\left(1-\left(\frac{\tau}{\sigma_0}\right)^{2(r-k)}\frac{\cosh 2\beta t}{\cosh 2\beta\sigma_0}\right)y(\tau)e^{i\tau t}\,d\tau$$

where $\sigma_0=\min\{\sigma,\widehat{\sigma}\}$ is optimal.

Proof. Consider the extremal problem

$$\|f^{(k)}(\cdot)\|^2_{L_2(\mathbb{R})}\to\max,\quad \|f^{(r)}(\cdot)\|^2_{\mathcal{H}^\beta_2}\le 1,\quad \|\widehat{f}(\cdot)\|^2_{L_\infty(\Delta_\sigma)}\le\delta^2. \tag{27}$$

For this problem the Lagrange function has the form

$$\mathcal{L}(f(\cdot),\lambda_1,\lambda_2(\cdot))=-\|f^{(k)}(\cdot)\|^2_{L_2(\mathbb{R})}+\lambda_1\|f^{(r)}(\cdot)\|^2_{\mathcal{H}^\beta_2}+\int_{\Delta_\sigma}\lambda_2(t)|\widehat{f}(t)|^2\,dt.$$

Passing to Fourier transforms and writing $(2\pi)^{-1}|\widehat{f}(\cdot)|^2=u(\cdot)$, we have

$$\mathcal{L}(f(\cdot),\lambda_1,\lambda_2(\cdot))=\int_{\mathbb{R}}(-t^{2k}+\lambda_1 t^{2r}\cosh 2\beta t)u(t)\,dt+2\pi\int_{\Delta_\sigma}\lambda_2(t)u(t)\,dt.$$

Set $\widehat{\lambda}_1 = \sigma_0^{-2(r-k)} \cosh^{-1} 2\beta\sigma_0$ and

$$\widehat{\lambda}_2(t) = \begin{cases} (2\pi)^{-1}\left(t^{2k} - \widehat{\lambda}_1 t^{2r} \cosh 2\beta t\right), & |t| < \sigma_0, \\ 0, & |t| \geq \sigma_0. \end{cases}$$

Then for all $f(\cdot) \in \mathcal{H}_{2,\infty}^{r,\beta}$

$$\mathcal{L}(f(\cdot), \widehat{\lambda}_1, \widehat{\lambda}_2(\cdot)) = \int_{|t|\geq\sigma_0} \left(-t^{2k} + \widehat{\lambda}_1 t^{2r} \cosh 2\beta t\right) u(t)\, dt \geq 0.$$

If $\sigma \geq \widehat{\sigma}$, then denoting by $g(\cdot)$ the inverse Fourier transform of the function which equals δ on the interval $(-\widehat{\sigma}, \widehat{\sigma})$ and vanishes outside, it is easy to verify that conditions (b) and (c) of Theorem 4 are fulfilled for the constant sequence $f_n(\cdot) = g(\cdot)$.

If $\sigma < \widehat{\sigma}$, we set

$$\gamma = 1 - \delta^2 I_{r,\beta}(\sigma).$$

Consider the sequence of functions

$$u_n(t) = \begin{cases} \dfrac{\delta^2}{2\pi}, & |t| < \sigma, \\ \dfrac{n\gamma}{2(\sigma + 1/n)^{2r} \cosh 2\beta(\sigma + 1/n)}, & \sigma \leq |t| \leq \sigma + 1/n, \\ 0, & |t| > \sigma + 1/n \end{cases}$$

(by $g_n(\cdot)$ we denote the inverse Fourier transforms of the functions $\sqrt{2\pi u_n(\cdot)}$). It is easy to verify that

$$\lim_{n\to\infty} \mathcal{L}(g_n(\cdot), \widehat{\lambda}_1, \widehat{\lambda}_2(\cdot)) = 0.$$

Moreover,

$$(28) \quad \|g_n^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta}^2 = \int_{\mathbb{R}} t^{2r} u_n(t) \cosh 2\beta t\, dt$$

$$= 1 - \gamma + \frac{n\gamma}{(\sigma + 1/n)^{2r} \cosh 2\beta(\sigma + 1/n)} \int_\sigma^{\sigma+1/n} t^{2r} \cosh 2\beta t\, dt < 1,$$

that is, the functions $g_n(\cdot)$ are admissible in the problem (27). It follows also from (28) that

$$\lim_{n\to\infty} \|g_n^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta} = 1.$$

Here the problem (7) has the form

$$\widehat{\lambda}_1 \|f^{(r)}(\cdot)\|_{\mathcal{H}_2^\beta} + \int_{\Delta_\sigma} \widehat{\lambda}_2(t) |\widehat{f}(t) - y(t)|^2\, dt \to \min, \quad f(\cdot) \in \mathcal{H}_{2,\infty}^{r,\beta}.$$

Passing to Fourier transforms it can be written in the form

$$\frac{\widehat{\lambda}_1}{2\pi} \int_{\mathbb{R}} t^{2r} |\widehat{f}(t)|^2 \cosh 2\beta t\, dt + \int_{\Delta_\sigma} \widehat{\lambda}_2(t) |\widehat{f}(t) - y(t)|^2\, dt \to \min,$$

$$f(\cdot) \in \mathcal{H}_{2,\infty}^{r,\beta}.$$

It is easy to verify that the function $f_y(\cdot)$ such that

$$\widehat{f}_y(t)=\begin{cases}\dfrac{2\pi\widehat{\lambda}_2(t)}{2\pi\widehat{\lambda}_2(t)+\widehat{\lambda}_2t^{2r}\cosh 2\beta t}y(t), & |t|<\sigma_0,\\ 0, & |t|\ge\sigma_0,\end{cases}$$

that is,

$$\widehat{f}_y(t)=\begin{cases}\left(1-\left(\dfrac{t}{\sigma_0}\right)^{2(r-k)}\dfrac{\cosh 2\beta t}{\cosh 2\beta\sigma_0}\right)y(t), & |t|<\sigma_0,\\ 0, & |t|\ge\sigma_0,\end{cases}$$

is the solution of this problem. It follows by Theorem 4 that the method

$$\varphi_0(y)(\cdot)=f_y^{(k)}(\cdot)$$

is optimal, and for its error the equality

$$E_k(H^{r,\beta}_{2,\infty},\delta,\sigma)=\sqrt{\widehat{\lambda}_1+\delta^2\int_{\Delta_\sigma}\widehat{\lambda}_2(t)\,dt}$$

holds. Substituting the expressions for $\widehat{\lambda}_1$ and $\widehat{\lambda}_2(\cdot)$, we obtain the statement of the theorem. □

It follows by Theorem 6 that for a given δ, starting from $\widehat{\sigma}$, further extension of the interval on which the Fourier transform of a function in $H^{r,\beta}_{2,\infty}$ is given with error δ in the uniform metric does not result in a decrease in the recovery error. In other words, if the relation

$$\frac{\delta^2}{2\pi}\int_{-\sigma}^{\sigma}t^{2r}\cosh 2\beta t\,dt\le 1 \tag{29}$$

between the error of input data and the size of the interval on which the data is measured is violated, then the available information turns out to be redundant.

4.3. **Theorem about three circles.** Denote by $\mathcal{H}_2$ the set of functions $f(\cdot)$ analytic in the unit disc $D=\{z\in\mathbb{C}:|z|<1\}$ for which

$$\sup_{0<r<1}\int_{\mathbb{T}}|f(re^{it})|^2\,dt<\infty.$$

Consider the analogue of theorem about three lines (Theorem 3) for functions from $\mathcal{H}_2$. We are interested in the extremal problem on finding the value

$$\sup_{\substack{f(\cdot)\in\mathcal{H}_2\\ \|f(r_1e^{i\cdot})\|_{L_2(\mathbb{T})}\le\delta_1\\ \|f(r_2e^{i\cdot})\|_{L_2(\mathbb{T})}\le\delta_2}}\|f(\rho e^{i\cdot})\|_{L_2(\mathbb{T})}$$

where $0<r_1<\rho<r_2\le 1$. We connect this problem with the problem of recovery of $f(\cdot)\in\mathcal{H}_2$ on the circle $|z|=\rho$ by approximate values of

this function on the circles $|z|=r_1$ and $|z|=r_2$. We assume that for every function $f(\cdot)\in\mathcal{H}_2$ we know $y_1(\cdot),y_2(\cdot)\in L_2(\mathbb{T})$ such that

$$\|f(r_je^{i\cdot})-y_j(\cdot)\|_{L_2(\mathbb{T})}\le\delta_j,\quad j=1,2.$$

The task is to recover $f(\rho e^{i\cdot})$ in the best way by functions $y_1(\cdot),y_2(\cdot)$.

The quantity

$$e_\rho(\mathcal{H}_2,\delta_1,\delta_2,\varphi)=\sup_{\substack{f(\cdot)\in\mathcal{H}_2,\ y_1(\cdot),y_2(\cdot)\in L_2(\mathbb{T})\\ \|f(r_je^{i\cdot})-y_j(\cdot)\|_{L_2(\mathbb{T})}\le\delta_j,\ j=1,2}}\|f(\rho e^{i\cdot})-\varphi(y_1,y_2)(\cdot)\|_{L_2(\mathbb{T})}$$

is called the error of the recovery method $\varphi\colon L_2(\mathbb{T})\times L_2(\mathbb{T})\to L_2(\mathbb{T})$. The quantity

$$E_\rho(\mathcal{H}_2,\delta_1,\delta_2)=\inf_{\varphi\colon L_2(\mathbb{T})\times L_2(\mathbb{T})\to L_2(\mathbb{T})}e_\rho(\mathcal{H}_2,\delta_1,\delta_2,\varphi)$$

is called the error of optimal recovery, and a method delivering this lower bound is called the optimal method.

Theorem 7. *Let $0<r_1<\rho<r_2\le1$ and $\delta_1,\delta_2>0$. Set*

$$\Delta_s=[(r_1/r_2)^{s+1},(r_1/r_2)^s),\quad s=0,1,\dots,$$

$$\mu_{s1}=\frac{r_2^2-\rho^2}{r_1^{2s}},\quad \mu_{s2}=\frac{\rho^2-r_1^2}{r_2^{2s}}. \tag{30}$$

Then

$$E_\rho(\mathcal{H}_2,\delta_1,\delta_2)=\sup_{\substack{f(\cdot)\in\mathcal{H}_2\\ \|f(r_1e^{i\cdot})\|_{L_2(\mathbb{T})}\le\delta_1\\ \|f(r_2e^{i\cdot})\|_{L_2(\mathbb{T})}\le\delta_2}}\|f(\rho e^{i\cdot})\|_{L_2(\mathbb{T})}$$
$$=\begin{cases}\dfrac{\rho^s}{\sqrt{r_2^2-r_1^2}}\sqrt{\delta_1^2\mu_{s1}+\delta_2^2\mu_{s2}} & \delta_1/\delta_2\in\Delta_s,\ s=0,1,\dots,\\ \delta_2, & \delta_1\ge\delta_2.\end{cases}$$

For $\delta_1/\delta_2\in\Delta_s$, $s=0,1,\dots$, the method

$$\varphi_0(y_1,y_2)(\cdot)=\mu_{s1}\mathcal{K}_{\rho,\delta_1,\delta_2}(\rho r_1e^{i\cdot})*y_1(\cdot)+\mu_{s2}\mathcal{K}_{\rho,\delta_1,\delta_2}(\rho r_2e^{i\cdot})*y_2(\cdot),$$

where

$$\mathcal{K}_{\rho,\delta_1,\delta_2}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\mu_{s1}r_1^{2k}+\mu_{s2}r_2^{2k}}, \tag{31}$$

is optimal. If $\delta_1\ge\delta_2$, then the method

$$\varphi_0(y_1,y_2)(t)=\frac{1}{2\pi}\int_{\mathbb{T}}\frac{r_2}{r_2-\rho e^{ik(t-u)}}y_2(u)\,du$$

is optimal.

Proof. We consider the extremal problem

$$\|f(\rho e^{i\cdot})\|^2_{L_2(\mathbb{T})} \to \max, \quad \|f(r_j e^{i\cdot}\|^2_{L_2(\mathbb{T})} \le \delta_j^2, \quad j=1,2, \tag{32}$$

and write the corresponding Lagrange function

$\mathcal{L}(f(\cdot),\lambda_1,\lambda_2) = -\|f(\rho e^{i\cdot})\|^2_{L_2(\mathbb{T})} + \lambda_1\|f(r_1e^{i\cdot})\|^2_{L_2(\mathbb{T})} + \lambda_2\|f(r_2e^{i\cdot})\|^2_{L_2(\mathbb{T})}$.

For $f(\cdot)\in\mathcal{H}_2$ which has the form

$$f(z)=\sum_{k=0}^{\infty}a_kz^k$$

for all $0<r\le 1$ the equality

$$\|f(re^{i\cdot})\|^2_{L_2(\mathbb{T})} = \sum_{k=0}^{\infty}|a_k|^2r^{2k}$$

holds. Therefore the Lagrange function can be written in the form

$$\mathcal{L}(f(\cdot),\lambda_1,\lambda_2)=\sum_{k=0}^{\infty}(-\rho^{2k}+\lambda_1r_1^{2k}+\lambda_2r_2^{2k})|a_k|^2.$$

For all $\lambda_1,\lambda_2\ge 0$ the function

$$\alpha(x) = -1+\lambda_1\left(\frac{r_1}{\rho}\right)^{2x}+\lambda_2\left(\frac{r_2}{\rho}\right)^{2x}$$

is convex. Consequently, if at some s

$$\alpha(s)=\alpha(s+1)=0, \tag{33}$$

then $\alpha(x)\ge 0$ for all $x\in[0,s]\cup[s+1,+\infty)$. Thus, if we take $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ from the condition (33), then for all $k\in\mathbb{Z}_+$

$$-\rho^{2k}+\widehat{\lambda}_1r_1^{2k}+\widehat{\lambda}_2r_2^{2k}=\rho^{2k}\alpha(k)\ge 0.$$

For $\widehat{\lambda}_1$ and $\widehat{\lambda}_2$ we have

$$\widehat{\lambda}_1=\frac{\rho^{2s}}{r_2^2-r_1^2}\mu_{s1}, \quad \widehat{\lambda}_2=\frac{\rho^{2s}}{r_2^2-r_1^2}\mu_{s2},$$

where μ_{s1} and μ_{s2} are defined by (30). Thus, for all $f(\cdot)\in\mathcal{H}_2$

$$\mathcal{L}(f(\cdot),\widehat{\lambda}_1,\widehat{\lambda}_2)\ge 0.$$

Let $\delta_1/\delta_2\in\Delta_s$. Set

$$g(z)=\widehat{a}_sz^s+\widehat{a}_{s+1}z^{s+1}$$

where

$$\widehat{a}_s=\left(\frac{\delta_1^2r_2^{2s+2}-\delta_2^2r_1^{2s+2}}{(r_1r_2)^{2s}(r_2^2-r_1^2)}\right)^{1/2}, \quad \widehat{a}_{s+1}=\left(\frac{\delta_2^2r_1^{2s}-\delta_1^2r_2^{2s}}{(r_1r_2)^{2s}(r_2^2-r_1^2)}\right)^{1/2}.$$

It is easy to verify that

$$\|g(r_1e^{i\cdot})\|_{L_2(\mathbb{T})}=\delta_1, \quad \|g(r_2e^{i\cdot})\|_{L_2(\mathbb{T})}=\delta_2$$

and $\mathcal{L}(g(\cdot), \widehat{\lambda}_1, \widehat{\lambda}_2) = 0$. It follows by Theorem 4 that the value of the extremal problem (32) is equal to

$$\widehat{\lambda}_1\delta_1^2 + \widehat{\lambda}_2\delta_2^2 = \frac{\rho^{2s}}{r_2^2 - r_1^2}(\delta_1^2\mu_{s1} + \delta_2^2\mu_{s2}).$$

Here the extremal problem (7) has the form

$$\|f(r_1e^{i\cdot}) - y_1(\cdot)\|^2_{L_2(\mathbb{T})} + \|f(r_2e^{i\cdot}) - y_2(\cdot)\|^2_{L_2(\mathbb{T})} \to \min, \quad f(\cdot) \in \mathcal{H}_2.$$

It is easy to verify that its solution is the function

$$f_{y_1,y_2}(z) = \sum_{k=0}^{\infty} \frac{\widehat{\lambda}_1 r_1^k (y_1)_k + \widehat{\lambda}_2 r_2^k (y_2)_k}{\widehat{\lambda}_1 r_1^{2k} + \widehat{\lambda}_2 r_2^{2k}} z^k,$$

where $(y_1)_k, (y_2)_k$, $k = 0, 1, \dots$, are the Fourier coefficients of $y_1(\cdot)$ and $y_2(\cdot)$, respectively. It follows by Theorem 4 that the method

$$\begin{aligned}\varphi_0(y_1, y_2)(\cdot) &= f_{y_1,y_2}(\rho e^{i\cdot}) \\ &= \mu_{s1}\mathcal{K}_{\rho,\delta_1,\delta_2}(\rho r_1 e^{i\cdot}) * y_1(\cdot) + \mu_{s2}\mathcal{K}_{\rho,\delta_1,\delta_2}(\rho r_2 e^{i\cdot}) * y_2(\cdot)\end{aligned}$$

is optimal.

For $\delta_1 \geq \delta_2$ we set $\widehat{\lambda}_1 = 0$, $\widehat{\lambda}_2 = 1$, $g(\cdot) = \delta_2$. In all other respects this case is considered analogously as before. □

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